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Exclusion of oscillations in heterogeneous and bi-composite media thermal conduction

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Abstract

Analysis of Fourier heat conduction in heterogeneous and bi-composite media (e.g. porous media, fluid suspensions, etc.) subject to Lack of Local Thermal Equilibrium (LaLotheq) reveals a condition for thermal oscillations and resonance to be possible. This paper shows that this condition cannot be fulfilled because of physical constraints leading to the exclusion of thermal waves and resonance.

Keywords: Heterogeneous media; Bi-composite media; Porous media conduction; Local thermal equilibrium; Local thermal non-equilibrium; Thermal waves; Lotheq; LaLotheq

1. Introduction

Previous analyses [1,2] showed that the physical conditions necessary for thermal waves to be possible in porous media heat conduction subject to Lack of Local thermal equilibrium (LaLotheq) cannot be fulfilled by a Dual-Phase-Lagging (DuPhlag) approximation of the two phase conduction process for a rectangular slab subject to a combination of Dirichlet–Dirichlet [1] or Dirichlet–Newmann [2] set of boundary conditions. The present paper demonstrates that for a combination of Dirichlet–Dirichlet boundary conditions the exclusion of oscillations and consequently resonance is anticipated in the general case and not only in the Dual-Phase-Lagging (DuPhlag) approximation limit. The results apply also not only to porous media but to any heterogeneous system consisting of two phases, such as fluid suspensions [3], or bi-composite materials consisting of a combination of two different solid phases. When both phases are interconnected the derivations presented below apply accurately, while for the case when one phase is continuous and the other is dispersed (such as solid particles suspended in a fluid) the Dual-Phase-Lagging (DuPhlag) formulation applies accurately and not merely as an approximation as demonstrated by Vadasz [3]. In the latter case the DuPhlag results presented by Vadasz [1,2] that are excluding thermal waves are also accurately applicable.

The system of governing equations for Fourier conduction in porous media subject to Lack of Local Thermal Equilibrium (LaLotheq) was showed by Tzou [4] to be approximately equivalent to the Dual-Phase-Lagging (DuPhlag) model of heat conduction. The latter can produce thermal waves in the form of oscillations (see [4]). As a result the Dual-Phase-Lagging (DuPhlag) model can yield thermal resonance when periodically forced by a periodic heat source or a periodic boundary condition with a forcing frequency that is in the neighbourhood of one of the natural frequencies of the system. Tzou [4–6] presents applications of the DuPhlag model to a wide variety of fields from ultrafast (femtosecond) pulse-laser heating of metal films, phonon–electron interaction at nano and micro-scale heat transfer, temperature pulses in superfluid

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Nomenclature

Bh	bi-harmonic dimensionless group, β_e/L^2
Bf	bi-harmonic-Fourier dimensionless group, Bh/Fo_q
c^2	speed of propagation of the thermal wave, τ_q/α_e (dimensional)
$c_{p,f}$, c_s	fluid and solid phase specific heat, respectively (dimensional)
c_n	dimensionless damping coefficient defined by Eq. (31)
Fo_q	heat flux related Fourier number, $\alpha_e\tau_q/L^2$
Fo_T	temperature gradient related Fourier number, $\alpha_e\tau_T/L^2$
h	integral heat transfer coefficient for the heat conduction at the solid–fluid interface (dimensional)
k_s	effective thermal conductivity of the solid phase, $(1-\varphi)\tilde{k}_s$ (dimensional)
$\tilde{k}_s$	thermal conductivity of the solid phase, (dimensional)
k_f	effective thermal conductivity of the fluid phase, $\varphi\tilde{k}_f$ (dimensional)
$\tilde{k}_f$	thermal conductivity of the fluid phase, (dimensional)
L	the length of the slab (dimensional)
$\boldsymbol{q}$	heat flux vector (dimensional)
t_*	time (dimensional)
T	temperature, (dimensional)
T_C	coldest wall temperature (dimensional)
T_H	hottest wall temperature (dimensional)
x_*	horizontal co-ordinate (dimensional)

Greek symbols

α_e	effective thermal diffusivity, defined by Eq. (5), (dimensional)
β_e	bi-harmonic coefficient, defined in Eq. (5) (dimensional)
γ_s	solid phase effective heat capacity, $(1-\varphi)\rho_s c_s$ (dimensional)
γ_f	fluid phase effective heat capacity, $\varphi\rho_f c_{p,f}$ (dimensional)
θ	dimensionless temperature, $(T-T_C)/(T_H-T_C)$
φ	porosity
ρ_s	solid phase density
ρ_f	fluid phase density
τ_q	time lag associated with the heat flux, defined by Eq. (5), (dimensional)
τ_T	time lag associated with the temperature gradient defined by Eq. (5), (dimensional)
ω_n	dimensionless natural thermal frequency defined by Eq. (31)
ψ	time lags ratio defined by Eq. (10)

Subscripts

*	corresponding to dimensional values of the independent variables, except for cases where there is no ambiguity, as listed in this nomenclature
s	related to the solid phase
f	related to the fluid phase

liquid helium, thermal lagging in amorphous materials, and thermal waves under rapidly propagating cracks.

Analytical solutions as well as analysis of the DuPhlag heat conduction were presented among others in excellent papers by Xu and Wang [7], Wang et al. [8], and Wang and Xu [9] and Antaki [10].

Applications of porous media heat transfer subject to Lack of Local Thermal Equilibrium (LaLotheq) were undertaken among others by Nield [11], Minkowycz et al. [12], Banu and Rees [13], Baytas and Pop [14], Kim and Jang [15], Rees [16], Alazmi and Vafai [17], and Nield et al. [18]. While the significance of practically obtaining the same temperature solution for each phase in a porous medium subject to a Lack of Local Thermal Equilibrium (LaLotheq) is discussed by Vadasz [19] identifying conditions for which the traditional formulation of the LaLotheq model might not be adequate, the conditions used in the present paper are not identical to those identified by Vadasz [19].

The present paper deals with Fourier heat conduction in a heterogeneous (e.g. porous) or bi-composite medium subject to LaLotheq. The latter produces a bi-harmonic linear partial differential equation that possesses wave properties. Nevertheless, physical constraints exclude the possibility of thermal wave solutions in such systems. The present paper demonstrates this exclusion for a heterogeneous (e.g. porous) or bi-composite slab subject to a combination of Dirichlet–Dirichlet boundary conditions.

2. Problem formulation and properties of the LaLotheq system

The following analysis uses the terminology applicable to heat conduction in porous media, although it applies equally well to any other heterogeneous or bi-composite system. Therefore the terminology of “solid phase–fluid phase” should be converted to “solid-phase 1–solid phase 2” in the case of bi-composite systems and similar conversions apply to other two-phase systems. The heat conduction equations for the two phases that compose an isotropic and homogeneous porous medium subject to LaLotheq are obtained as phase averages over a Representative Elementary Volume (REV) following *Fourier’s Law* in the form:

$$\gamma_s \frac{\partial T_s}{\partial t_*} = k_s \nabla_*^2 T_s - h(T_s - T_f), \tag{1}$$

$$\gamma_f \frac{\partial T_f}{\partial t_*} = k_f \nabla_*^2 T_f + h(T_s - T_f), \tag{2}$$

where $\gamma_s = (1-\varphi)\rho_s c_s$ and $\gamma_f = \varphi\rho_f c_{p,f}$ are the solid phase and fluid phase effective heat capacities, respectively, φ is the porosity, k_s and k_f are the effective thermal conductivities of the solid and fluid phases, respectively, and h represents an integral heat transfer coefficient for the heat conduction at the solid–fluid interface within an REV, assumed to be independent of time and anticipated to depend on the thermal conductivities of both phases, on the porosity, on the heat transfer surface area and on the tortuousity of the interface between the solid and fluid phases [20,21]. In the case of fluid flow the value of h will depend also on local Reynolds and Prandtl numbers of the fluid as presented by Alazmi and Vafai [17]. In the case of solid particles suspended in a fluid the effective thermal conductivity of the solid phase vanishes, i.e. $k_s = 0$, leading to lack of macroscopic level conduction heat transfer within the solid phase because the solid particles represent the dispersed phase in the fluid suspension and therefore the solid particles can conduct heat between themselves only via the neighbouring fluid (see [3] for details).

When the Local Thermal Equilibrium (Lotheq) assumption is not valid, conditions appropriate for the case when the temperature difference between the two phases is not small, the two Eqs. (1) and (2) are to be solved simultaneously. The diffusion terms in these equations are a result of replacing the $-\nabla_* \cdot \boldsymbol{q}_s$ and $-\nabla_* \cdot \boldsymbol{q}_f$ terms by using Fourier's Law in the form $\boldsymbol{q}_s = -k_s\nabla_* T_s$ and $\boldsymbol{q}_f = -k_f \nabla_* T_f$ to yield the Laplacian terms. The coupling between the two equations can be resolved as presented by Vadasz [1–3,19] leading to

$$\left[\left(\gamma_s \frac{\partial}{\partial t_*} - k_s \nabla_*^2 + h\right)\left(\gamma_f \frac{\partial}{\partial t_*} - k_f \nabla_*^2 + h\right) - h^2\right] T_j = 0 \quad \forall j = \mathrm{s}, \mathrm{f}, \tag{3}$$

where the index j can take the values s representing the solid phase or f standing for the fluid phase. The explicit form of Eq. (3) is obtained after dividing it by $h(\gamma_s + \gamma_f)$ in the form

$$\tau_q \frac{\partial^2 T_j}{\partial t_*^2} + \frac{\partial T_j}{\partial t_*} = \alpha_e \left[\nabla_*^2 T_j + \tau_T \nabla_*^2 \left(\frac{\partial T_j}{\partial t_*}\right) - \beta_e \nabla_*^4 T_j\right] \quad \forall j = \mathrm{s}, \mathrm{f}, \tag{4}$$

where the following notation was used:

$$\tau_q = \frac{\gamma_s \gamma_f}{h(\gamma_s + \gamma_f)}; \quad \alpha_e = \frac{(k_s + k_f)}{(\gamma_s + \gamma_f)}; \quad \tau_T = \frac{(\gamma_s k_f + \gamma_f k_s)}{h(k_s + k_f)}; \quad \beta_e = \frac{k_s k_f}{h(k_s + k_f)}. \tag{5}$$

In Eqs. (4) and (5) τ_q and τ_T are the heat flux and temperature related time lags linked to the two-phase conduction delays due to the finite heat capacity of both phases, to be discussed below, while α_e is the effective thermal diffusivity of the porous medium. It may be observed from Eq. (5) that there is a dual effect of the heat capacities on the effective parameters of the uncoupled system in the sense that the heat flux time lag τ_q is affected by the heat capacities of the solid and fluid phases as thermal capacitors connected in series following the relationship:

$$\frac{1}{\gamma_e^s} = \frac{1}{\gamma_s} + \frac{1}{\gamma_f} = \frac{(\gamma_s + \gamma_f)}{\gamma_s\gamma_f} \to \gamma_e^s = \frac{\gamma_s \gamma_f}{(\gamma_s + \gamma_f)}, \tag{6}$$

while the effective thermal diffusivity α_e is affected by the heat capacities of the solid and fluid phases as thermal capacitors connected in parallel following the relationship:

$$\gamma_e^p = (\gamma_s + \gamma_f). \tag{7}$$

In addition the bi-harmonic parameter β_e can be presented as the ratio between the effective thermal conductivity due to the thermal resistances of the solid and fluid phases connected in series and the heat transfer coefficient h, in the form $\beta_e = k_e/h$, where

$$k_e^s = \frac{k_s k_f}{(k_s + k_f)} \tag{8}$$

and the thermal resistance of each phase is defined as $1/k_j$ $\forall j = \mathrm{s}, \mathrm{f}$.

Eq. (4) is the conduction equation for each phase of a porous medium that degenerates to the thermal diffusion equation when $\tau_q = \tau_T = \beta_e = 0$. The latter may occur either when the effective heat capacities and effective thermal conductivities of the solid and fluid phases are excessively low, i.e. when $(\gamma_s, \gamma_f, k_s, k_f) \to 0$, or when the fluid–solid interface heat transfer coefficient is excessively large, i.e. when $h \to \infty$ as can be observed from Eq. (5). When only the bi-harmonic term in Eq. (4) is negligibly small, or identically vanishes i.e. if $\beta_e = 0$, the equation transforms into the Dual-Phase-Lagging equation. The latter applies accurately to fluid suspensions where $k_s = 0$ as discussed in the text following Eq. (2), or approximately to heterogeneous and bi-composite media when β_e is very small but not identically zero.

The wave properties of Eq. (4) can be observed after dividing it by α_e to obtain

$$\frac{1}{c^2}\frac{\partial^2 T_j}{\partial t_*^2} + \frac{1}{\alpha_e}\frac{\partial T_j}{\partial t_*} = \nabla_*^2 T_j + \tau_T \nabla_*^2\left(\frac{\partial T_j}{\partial t_*}\right) - \beta_e \nabla_*^4 T_j \quad \forall j = \mathrm{s}, \mathrm{f} \tag{9}$$

where $c^2 = \alpha_e/\tau_q$ is the accepted definition of the speed of propagation of the thermal wave.

A direct property of the parameters defined in Eq. (5) is obtained by evaluating the ratio τ_T/τ_q, which leads to the following result:

$$\psi = \frac{\tau_T}{\tau_q} = 1 + \frac{\gamma_s^2 k_f + \gamma_f^2 k_s}{\gamma_s \gamma_f (k_s + k_f)} > 1. \tag{10}$$

Since the combination of positive valued properties in the second term of Eq. (10) is always positive, the time lags ratio is always greater than 1, i.e. $\tau_T/\tau_q > 1$. The latter conclusion is based on a physical argument and it is accurately derived. It applies generally to Fourier heat conduction in heterogeneous media subject to LaLotheq and is not restricted to any specific geometry, nor boundary conditions. Note that while each one of the time lags τ_T and τ_q depend on the interface heat transfer coefficient h as observed in Eq. (5), their ratio τ_T/τ_q in Eq. (10) is independent of this coefficient making its evaluation simpler as it depends on the effective properties of each phase and is independent of the interaction between the phases.

Further analysis is used now to derive an additional inequality involving the time lags ratio ψ that will be useful in the following derivations and subsequently on the conclusions. For the following analysis it is convenient to introduce the following notation for the effective thermal conductivities ratio and effective heat capacities ratio in the form:

$$\eta_k = \frac{k_{\mathrm{f}}}{k_{\mathrm{s}}}; \quad \eta_\gamma = \frac{\gamma_{\mathrm{f}}}{\gamma_{\mathrm{s}}}. \tag{11}$$

Then the time lags ratio ψ defined in Eq. (10) can be expressed in terms of η_k and η_γ in the form

$$\psi = 1 + \frac{\eta_\gamma}{(1+\eta_k)} + \frac{\eta_\gamma^{-1}}{(1+\eta_k^{-1})}. \tag{12}$$

It can be easily checked that Eq. (12) is identical to the definition of ψ from (10) by substituting Eq. (11) into (12) followed by some algebra to produce the expression given in the definition of ψ in Eq. (10). An additional dimensionless ratio Bf that will prove useful in the following analysis and subsequent derivations is introduced now

$$\mathrm{Bf} = \frac{k_{\mathrm{s}}k_{\mathrm{f}}(\gamma_{\mathrm{s}}+\gamma_{\mathrm{f}})^2}{(k_{\mathrm{s}}+k_{\mathrm{f}})^2\gamma_{\mathrm{s}}\gamma_{\mathrm{f}}} = \frac{k_{\mathrm{e}}^{\mathrm{s}}}{k_{\mathrm{e}}^{p}}\frac{\gamma_{\mathrm{e}}^{p}}{\gamma_{\mathrm{e}}^{\mathrm{s}}} = \frac{2+\eta_\gamma^{-1}+\eta_\gamma}{2+\eta_k^{-1}+\eta_k}. \tag{13}$$

The following identities are also useful:

$$(1+\eta_k)(1+\eta_k^{-1}) = 2+\eta_k^{-1}+\eta_k, \tag{14}$$

$$(1+\eta_\gamma)(1+\eta_\gamma^{-1}) = 2+\eta_\gamma^{-1}+\eta_\gamma. \tag{15}$$

By using identities (14) and (15) an alternative form of Eq. (13) is obtained

$$\mathrm{Bf} = \frac{2+\eta_\gamma^{-1}+\eta_\gamma}{2+\eta_k^{-1}+\eta_k} = \frac{(1+\eta_\gamma)(1+\eta_\gamma^{-1})}{(1+\eta_k)(1+\eta_k^{-1})}. \tag{16}$$

Now we create a common denominator for the last two terms in Eq. (12) leading to

$$\begin{aligned}\psi &= 1 + \frac{\eta_\gamma}{(1+\eta_k)} + \frac{\eta_\gamma^{-1}}{(1+\eta_k^{-1})} \\ &= 1 + \frac{\eta_\gamma + \eta_\gamma^{-1} + \eta_\gamma^{-1}\eta_k + \eta_\gamma\eta_k^{-1}}{(1+\eta_k)(1+\eta_k^{-1})} = 1 + \frac{\eta_\gamma+\eta_\gamma^{-1}+\frac{(\eta_\gamma^2+\eta_k^2)}{\eta_\gamma\eta_k}}{(1+\eta_k)(1+\eta_k^{-1})} \\ &= \frac{\eta_\gamma+\eta_\gamma^{-1}+2+\frac{(\eta_\gamma-\eta_k)^2}{\eta_\gamma\eta_k}}{(1+\eta_k)(1+\eta_k^{-1})} \\ &= 1 + \frac{2+\eta_\gamma+\eta_\gamma^{-1}}{(1+\eta_k)(1+\eta_k^{-1})} + \frac{(\eta_\gamma-\eta_k)^2}{\eta_\gamma\eta_k(1+\eta_k)(1+\eta_k^{-1})}.\end{aligned}$$

By using identity (14) in the denominator of the second term above yields

$$\psi = 1 + \frac{2+\eta_\gamma+\eta_\gamma^{-1}}{2+\eta_k+\eta_k^{-1}} + \frac{(\eta_\gamma-\eta_k)^2}{\eta_\gamma\eta_k(1+\eta_k)(1+\eta_k^{-1})}. \tag{17}$$

One can recognize now that the second term in Eq. (17) is the Bf number defined in Eq. (13), transforming Eq. (17) into the following form:

$$\psi = 1 + \mathrm{Bf} + \frac{(\eta_\gamma-\eta_k)^2}{\eta_\gamma\eta_k(1+\eta_k)(1+\eta_k^{-1})}. \tag{18}$$

The third term in Eq. (18) is always non-negative, therefore the following inequality applies for the time lags ratio ψ:

$$\psi \geqslant 1 + \mathrm{Bf} \tag{19}$$

with the equality holding only for $\eta_k = \eta_\gamma$, leading to $\mathrm{Bf} = 1$ and $\psi = 2$. The latter conclusion expressed by inequality (19) is based on physical properties of the materials and has a profound impact on the following results.

The analysis of the bi-harmonic Eq. (4) governing heterogeneous and bi-composite media conduction subject to LaLotheq is undertaken for the particular case corresponding to the one dimensional heat conduction in a slab of length L as presented in Fig. 1. Transforming Eq. (4) into a dimensionless form by using L to scale the independent length variable x_*, i.e. $x = x_*/L$, by using L^2/α_{e} to scale the time, i.e. $t = \alpha_{\mathrm{e}}t_*/L^2$ and introducing the dimensionless temperature, θ_j

$$\theta_j = \frac{(T_j - T_{\mathrm{C}})}{(T_{\mathrm{H}} - T_{\mathrm{C}})} \quad \forall j = \mathrm{s}, \mathrm{f}, \tag{20}$$

where T_{C} and T_{H} are the cold-wall and hot-wall imposed temperatures, respectively, both considered constant yields

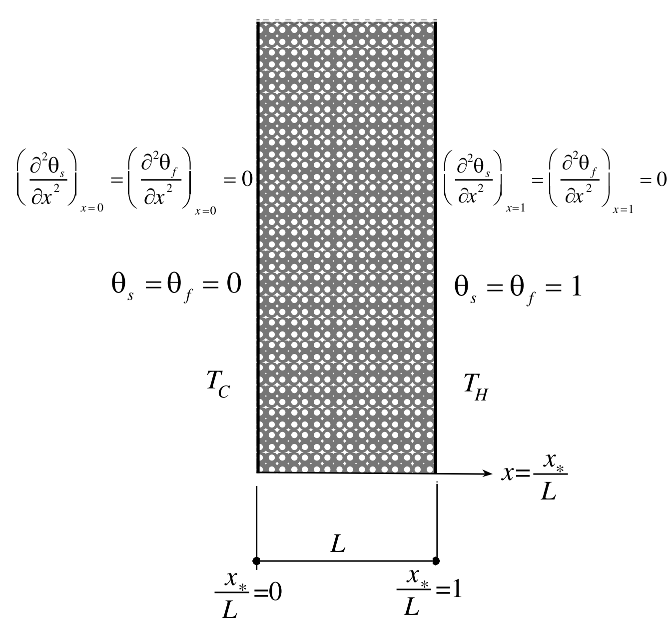

Fig. 1. A fluid saturated porous (or a bi-composite) slab subject to constant temperature conditions at the walls.

$$Fo_q \frac{\partial^2 \theta_j}{\partial t^2} + \frac{\partial \theta_j}{\partial t} = \frac{\partial^2 \theta_j}{\partial x^2} + Fo_T \frac{\partial^3 \theta_j}{\partial t \partial x^2} - \mathrm{Bh} \frac{\partial^4 \theta_j}{\partial x^4} \quad \forall j = \mathrm{s, f}, \tag{21}$$

where two Fourier numbers, Fo_q, Fo_T, and one additional dimensionless group, the bi-harmonic number Bh, emerged, and are defined in the form

$$Fo_q = \frac{\alpha_e \tau_q}{L^2}; \quad Fo_T = \frac{\alpha_e \tau_T}{L^2}; \quad \mathrm{Bh} = \frac{\beta_e}{L^2}. \tag{22}$$

Note that from Eq. (22) the form of the condition for the time lags ratio (10) remains the same when expressed in terms of the Fourier numbers ratio, i.e. $\psi = \tau_T/\tau_q = Fo_T/Fo_q > 1$. Similarly inequality (19) can be expressed in terms of the Fourier numbers ratio, i.e. $\psi = Fo_T/Fo_q \geqslant 1 + \mathrm{Bf}$.

We need four boundary conditions to be consistent with the differential equation, which is fourth order in the spatial coordinate. Therefore for a combination of Dirichlet–Dirichlet boundary conditions for Eqs. (1) and (2) the corresponding boundary conditions for Eq. (21) are

$$x = 0: \quad \theta_j = 0, \quad \left(\frac{\partial^2 \theta_j}{\partial x^2}\right)_{x=0} = 0 \quad \forall j = \mathrm{s, f}, \tag{23}$$

$$x = 1: \quad \theta_j = 1, \quad \left(\frac{\partial^2 \theta_j}{\partial x^2}\right)_{x=0} = 0 \quad \forall j = \mathrm{s, f}. \tag{24}$$

The second derivative boundary conditions in (23) and (24) are obtained by using Eqs. (1) and (2) in one dimension and by substituting the following conditions at the boundaries $(T_s)_{x_*=0} = (T_f)_{x_*=0} = T_C = \text{const.}$ and $(T_s)_{x_*=L} = (T_f)_{x_*=L} = T_H = \text{const.}$ leading to $(\partial^2 T_f/\partial x^2)_{x_*=0} = (\partial^2 T_s/\partial x^2)_{x_*=0} = 0$ and $(\partial^2 T_f/\partial x^2)_{x_*=L} = (\partial^2 T_s/\partial x^2)_{x_*=L} = 0$. The latter expressions are presented in a dimensionless form by the second derivative terms in Eqs. (23) and (24).

We need two initial conditions to be consistent with the differential equation, which is second order in time. The initial conditions applicable for initial distinct constant temperatures for the phases is

$$t = 0: \quad \theta_j = \theta_{j,o} = \text{const.} \quad \text{and} \quad \dot{\theta}_j = \dot{\theta}_{j,o} = \text{const.} \quad \forall j = \mathrm{s, f}, \tag{25}$$

where the dot symbol $(\cdot)$ is the Newtonian representation of a time derivative, and where the derivative initial condition was obtained by substituting the first constant temperature initial condition $(T_s)_{t_*=0} = T_{s,o} = \text{const.}$ and $(T_f)_{t_*=0} = T_{f,o} = \text{const.}$ into Eqs. (1) and (2) leading to $(\partial T_s/\partial t)_{t_*=0} = h(T_{s,o} - T_{f,o})/\gamma_s = \dot{T}_{s,o}$ and $(\partial T_f/\partial t)_{t_*=0} = h(T_{f,o} - T_{s,o})/\gamma_f = \dot{T}_{f,o}$. The latter is expressed in the following dimensionless form $\dot{\theta}_s = \dot{\theta}_{s,o} = \text{const.}$ and $\dot{\theta}_f = \dot{\theta}_{f,o} = \text{const.}$ as presented in Eq. (25). Note that the conclusions presented in this paper are independent of the initial conditions. These conditions affect only the coefficients of the final form of the series solution but not its eigenvalues and eigenfunctions. Any form of more general initial conditions lead to the same conclusions; the choice adopted here is therefore for simplicity only.

3. Analytical solution

As the equations, boundary and initial conditions for each phase are similar (though different due to the different constant values of the initial temperature time derivative), their solution will be qualitatively similar and we may drop the subscripts s and f for convenience, except when they appear in the initial conditions. The solution to Eq. (21) is separated into steady state θ_{sts} and transient θ_{tr} parts in the form $\theta = \theta_{sts} + \theta_{tr}$. The steady state is represented by the linear solution $\theta_{sts} = x$, which satisfies the boundary conditions Eqs. (23) and (24). Its derivation is a bit longer than the trivial derivation for the usual second order steady state conduction equation but eventually leads to the same solution. The transient solution θ_{tr} has to fulfil the equation

$$Fo_q \frac{\partial^2 \theta_{tr}}{\partial t^2} + \frac{\partial \theta_{tr}}{\partial t} = \frac{\partial^2 \theta_{tr}}{\partial x^2} + Fo_T \frac{\partial^3 \theta_{tr}}{\partial t \partial x^2} - \mathrm{Bh} \frac{\partial^4 \theta_{tr}}{\partial x^4} \tag{26}$$

and the following boundary and initial conditions:

$$\text{at } x = 0 \text{ and } x = 1: \quad (\theta_{tr})_{x=0,1} = 0, \quad \left(\frac{\partial^2 \theta_{tr}}{\partial x^2}\right)_{x=0,1} = 0, \tag{27}$$

$$t = 0: \quad \theta_{j,tr} = (\theta_{j,o} - x) \quad \text{and} \quad \dot{\theta}_{j,(tr)} = \dot{\theta}_{j,o} \quad \forall j = \mathrm{s, f}. \tag{28}$$

The solution is obtained by separation of variables in the form of two equations for each phase $\theta_{tr} = \phi_n(t) u_n(x)$, where the functions $\phi_n(t)$ and $u_n(x)$ are identical for both phases because of the identical boundary conditions (14) and (15). These equations are

$$\frac{d^2 \phi_n}{dt^2} + c_n \frac{d\phi_n}{dt} + \omega_n^2 \phi_n = 0, \tag{29}$$

$$\frac{d^2 u_n}{dx^2} + \kappa_n^2 u_n = 0. \tag{30}$$

The solution of Eq. (30) subject to the homogeneous boundary conditions $(u_n)_{x=0,1} = 0$ and $(d^2 u_n/dx^2)_{x=0,1} = 0$ at $x = 0,1$ is $u_n = a_n \sin(\kappa_n x)$ and the resulting eigenvalues are $\kappa_n = n\pi$, $\forall n = 0, 1, 2, 3, \ldots$ The coefficients c_n and ω_n^2 in Eq. (29) are defined in the form

$$c_n = Fo_q^{-1}(1 + \kappa_n^2 Fo_T) = Fo_q^{-1}(1 + n^2\pi^2 Fo_T), \tag{31}$$

$$\omega_n^2 = Fo_q^{-1} \kappa_n^2 (1 + \kappa_n^2 \mathrm{Bh}) = Fo_q^{-1} n^2 \pi^2 (1 + n^2 \pi^2 \mathrm{Bh}). \tag{32}$$

Eq. (29) represents a linear damped oscillator having a natural frequency ω_n and a damping coefficient c_n. Its eigenvalues are

$$\lambda_{1n} = -\frac{c_n}{2}\left[1 + \sqrt{1 - 4\frac{\omega_n^2}{c_n^2}}\right], \tag{33}$$

$$\lambda_{2n} = -\frac{c_n}{2}\left[1 - \sqrt{1 - 4\frac{\omega_n^2}{c_n^2}}\right]. \tag{34}$$

The solution for ϕ_n is overdamped if for some values of n the condition $4\omega_n^2 < c_n^2$ is satisfied, leading to

$$\theta_{tr,n} = (A_{1n} e^{\lambda_{1n} t} + A_{2n} e^{\lambda_{2n} t}) \sin(\kappa_n x), \tag{35}$$

it is critically-damped if for some values of $n = n_{cr}$ the condition $4\omega_{n_{cr}}^2 = c_{n_{cr}}^2$ is satisfied, i.e. $\lambda_{1n} = \lambda_{2n} = \lambda_{n_{cr}} = -c_{n_{cr}}/2$ leading to

$$\theta_{tr,n_{cr}} = (A_{1n_{cr}} e^{\lambda_{n_{cr}} t} + A_{2n_{cr}} t e^{\lambda_{n_{cr}} t}) \sin(\kappa_{n_{cr}} x) \tag{36}$$

and it is underdamped if for some values of n the condition $4\omega_n^2 > c_n^2$ is satisfied, i.e. $\lambda_{1n} = \lambda_{rn} - i\lambda_{in}$ and $\lambda_{2n} = \lambda_{rn} + i\lambda_{in}$, where $\lambda_{rn} = -c_n/2$ and $\lambda_{in} = \sqrt{4\omega_n^2 - c_n^2}/2$, leading to decaying thermal waves in the form

$$\theta_{tr,n} = e^{-\frac{c_n}{2}t}\{A_{1n}[\cos(\lambda_{in}t - \kappa_n x) - \cos(\lambda_{in}t + \kappa_n x)] - A_{2n}[\sin(\lambda_{in}t - \kappa_n x) - \sin(\lambda_{in}t + \kappa_n x)]\}. \tag{37}$$

In general the complete solution assuming the existence of one and only one critical value of $n_{cr} \geqslant 1$ has the form

$$\begin{aligned}\theta = x &+ \sum_{n=1}^{N_o}[A_n e^{\lambda_{n1}t} + B_n e^{\lambda_{n2}t}]\sin(n\pi x) \\ &+ e^{\lambda_{n_{cr}}t}[A_{n_{cr}} + B_{n_{cr}}t]\sin(n_{cr}\pi x)\delta_{n_{cr},m} \\ &+ \sum_{n=N_1}^{\infty} e^{-\frac{c_n}{2}t}\{A_n[\cos(\lambda_{in}t - \kappa_n x) - \cos(\lambda_{in}t + \kappa_n x)] \\ &- B_n[\sin(\lambda_{in}t - \kappa_n x) - \sin(\lambda_{in}t + \kappa_n x)]\},\end{aligned} \tag{38}$$

where

$$N_o = \lfloor n_{cr} \rfloor - \delta_{n_{cr},m} = \begin{cases}(n_{cr}+1) & \forall n_{cr} = m, \quad m = 1,2,3,\ldots, \\ \lfloor n_{cr} \rfloor & \forall n_{cr} \neq m, \quad m = 1,2,3,\ldots,\end{cases} \tag{39}$$

$$N_1 = \lfloor n_{cr} \rfloor + 1 = \begin{cases}(n_{cr}+1) & \forall n_{cr} = m, \quad m = 1,2,3,\ldots, \\ N_o + 1 & \forall n_{cr} \neq m, \quad m = 1,2,3,\ldots,\end{cases} \tag{40}$$

where $\delta_{n_{cr},m}$ is the Kronecker delta function defined in the form

$$\delta_{n_{cr},m} = \begin{cases}1 & \forall n_{cr} = m, \quad m = 1,2,3,\ldots, \\ 0 & \forall n_{cr} \neq m, \quad m = 1,2,3,\ldots\end{cases} \tag{41}$$

and where $\lfloor n_{cr} \rfloor$ is the inclusive floor function representing the largest integer less than or equal to n_{cr}. The value of n_{cr} is established from the condition $4\omega_{n_{cr}}^2 = c_{n_{cr}}^2$ and is expressed in the form

$$n_{cr} = \sqrt{-\frac{(\psi - 2)}{\pi^2 Fo_q(\psi^2 - 4\mathrm{Bf})}\left[1 \pm \frac{2}{(\psi - 2)}\sqrt{1 + \mathrm{Bf} - \psi}\right]}, \tag{42}$$

where $\psi = Fo_T/Fo_q$ is a dimensionless number representing the time lags ratio defined in Eq. (10) and $\mathrm{Bf} = \mathrm{Bh}/Fo_q$ is identical to the dimensionless group defined in Eq. (13) or (16) representing the ratio between the bi-harmonic number and the heat flux related Fourier number, Fo_q.

4. Conditions for oscillations and resonance

The condition for underdamped solutions and their associated oscillations is further explored to obtain explicit criteria in terms of the primitive parameters of the original system. The condition for underdamped (oscillatory) and critically damped solutions is obtained by using the definitions from Eqs. (31) and (32) and the conditions listed above following Eqs. (36) and (35), respectively, in the form:

$$\frac{c_n^2}{4\omega_n^2} = \frac{[1 + \kappa_n^2 Fo_T]^2}{4Fo_q\kappa_n^2(1 + \kappa_n^2\mathrm{Bh})} \leqslant 1, \tag{43}$$

where the inequality applies to underdamped conditions, while the equality corresponds to critically damped conditions. Inequality (43) can be expanded in the form

$$(Fo_T^2 - 4Fo_q\mathrm{Bh})\kappa_n^4 + 2(Fo_T - 2Fo_q)\kappa_n^2 + 1 \leqslant 0, \tag{44}$$

or alternatively in the form

$$Fo_q^2(\psi^2 - 4\mathrm{Bf})\kappa_n^4 + 2Fo_q(\psi - 2)\kappa_n^2 + 1 \leqslant 0. \tag{45}$$

We aim at deriving explicit conditions in terms of the primitive parameters of the system from inequality (45). To move towards this aim one may present inequality (45) in the form of a set of two inequalities

$$\begin{cases}\kappa_n^4 + \frac{2(\psi-2)}{Fo_q(\psi^2-4\mathrm{Bf})}\kappa_n^2 + \frac{1}{Fo_q^2(\psi^2-4\mathrm{Bf})} \leqslant 0 \quad \text{and} \quad \psi^2 > 4\mathrm{Bf}, \\ \qquad\qquad \text{or} \\ \kappa_n^4 + \frac{2(\psi-2)}{Fo_q(\psi^2-4\mathrm{Bf})}\kappa_n^2 + \frac{1}{Fo_q^2(\psi^2-4\mathrm{Bf})} \geqslant 0 \quad \text{and} \quad \psi^2 < 4\mathrm{Bf}.\end{cases} \tag{46}$$

These inequalities representing the conditions for underdamped and critically damped solutions may be presented in the form

$$\begin{cases}\kappa_n^4 + b\kappa_n^2 + c \leqslant 0 \quad \text{and} \quad \psi^2 > 4\mathrm{Bf}, \\ \qquad \text{or} \\ \kappa_n^4 + b\kappa_n^2 + c \geqslant 0 \quad \text{and} \quad \psi^2 < 4\mathrm{Bf},\end{cases} \tag{47}$$

where

$$b = \frac{2(\psi - 2)}{Fo_q(\psi^2 - 4\mathrm{Bf})}; \quad c = \frac{1}{Fo_q^2(\psi^2 - 4\mathrm{Bf})}. \tag{48}$$

We are looking for the domain of positive and negative values of the function

$$y \equiv \kappa_n^4 + b\kappa_n^2 + c. \tag{49}$$

The roots of this function, i.e. the intersection of the function with the κ_n^2 axis, represent the critical values of κ_n^2, i.e. $\kappa_{n,cr}^2$. They are obtained by equating y in Eq. (49) to zero, to yield the solution to the quadratic equation $y \equiv \kappa_n^4 + b\kappa_n^2 + c = 0$ in the form

$$\kappa_{n,cr}^2 = -\frac{(\psi - 2)}{Fo_q(\psi^2 - 4\mathrm{Bf})}\left[1 \pm \frac{2}{(\psi - 2)}\sqrt{1 + \mathrm{Bf} - \psi}\right]. \tag{50}$$

Since $\kappa_{n,cr}^2 = n^2\pi^2$ for $n = 1,2,3,\ldots$, the roots of the function y, presented in Eq. (50), have to be real leading to the condition $\psi < 1 + \mathrm{Bf}$. In addition, the roots $\kappa_{n,cr}^2$ have to be positive in order for $\kappa_{n,cr} = n_{cr}\pi$ to be real. The condition that the roots $\kappa_{n,cr}^2$ be real implies a condition that discriminant in Eq. (50) has to be non-negative, i.e. $(1 + \mathrm{Bf} - \psi) \geqslant 0$ or in the form

$$\psi \leqslant 1 + \mathrm{Bf}. \tag{51}$$

However, it was demonstrated in Eq. (19) that based on the physical properties of the materials $\psi \geqslant 1 + \mathrm{Bf}$ always, with the equality holding for $\eta_k = \eta_\gamma$ (corresponding to $\mathrm{Bf} = 1$ and $\psi = 2$). Since the condition (51) for real roots of $\kappa^2_{n,\mathrm{cr}}$ is in definite contrast with the physical reality expressed by inequality (19) one may conclude that underdamped oscillations and consequently thermal waves are excluded from the solution. The particular equality case of (19) might overlap with the equality in (51) but produces a degenerated and singular result. Therefore, by excluding these underdamped and critically damped modes transforms the solution (38) into the following form:

$$\theta = x + \sum_{n=1}^{\infty} [A_n \mathrm{e}^{\lambda_{n1} t} + B_n \mathrm{e}^{\lambda_{n2} t}] \sin(n\pi x), \tag{52}$$

where λ_{1n} and λ_{2n} are evaluated from Eqs. (33) and (34) and the coefficients A_n and B_n are being evaluated from the initial conditions. Clearly this is a solution of "hyper-diffusion" for the time variation, i.e. bi-exponential decay in time towards a time-independent steady state.

5. Conclusions

The heat conduction in heterogeneous and bi-composite media subject to Lack of Local Thermal Equilibrium (LaLotheq) lead to the expectation that thermal waves and resonance are possible. It was demonstrated that such thermal oscillations and consequently resonance are excluded in a slab subject to a combination of Dirichlet–Dirichlet boundary conditions.

References

[1] P. Vadasz, Absence of oscillations and resonance in porous media dual-phase-lagging Fourier heat conduction, J. Heat Transfer 127 (2004) 307–314.
[2] P. Vadasz, Lack of oscillations in dual-phase-lagging heat conduction for a porous slab subject to imposed heat flux and temperature, Int. J. Heat Mass Transfer 48 (2005) 2822–2828.
[3] P. Vadasz, Heat conduction in nanofluid suspensions, J. Heat Transfer 128 (2006) 465–477.
[4] D.Y. Tzou, Macro-to-Microscale Heat Transfer; The Lagging Behavior, Taylor & Francis, Washington DC, 1997.
[5] D.Y. Tzou, A unified field approach for heat conduction from macro-to-micro-scales, ASME J. Heat Transfer 117 (1995) 8–16.
[6] D.Y. Tzou, Temperature-dependent thermal lagging in ultrafast laser heating, Int. J. Heat Mass Transfer 44 (2001) 1725–1734.
[7] M. Xu, L. Wang, Thermal oscillation and resonance in dual-phase-lagging heat conduction, Int. J. Heat Mass Transfer 45 (2002) 1055–1061.
[8] L. Wang, M. Xu, X. Zhou, Well-posedness and solution structure of dual-phase-lagging heat conduction, Int. J. Heat Mass Transfer 44 (2001) 1659–1669.
[9] L. Wang, M. Xu, Well-posedness of dual-phase-lagging heat conduction equation: higher dimensions, Int. J. Heat Mass Transfer 45 (2002) 1165–1171.
[10] P.J. Antaki, Solution for non-fourier dual-phase-lag heat conduction in a semi-infinite slab with surface heat flux, Int. J. Heat Mass Transfer 41 (1998) 2253–2258.
[11] D.A. Nield, A note on the modeling of local thermal non-equilibrium in a structured porous medium, Int. J. Heat Mass Transfer 45 (2002) 4367–4368.
[12] W.J. Minkowycz, A. Haji-Shiekh, K. Vafai, On departure from local thermal equilibrium in porous media due to a rapidly changing heat source: the sparrow number, Int. J. Heat Mass Transfer 42 (1999) 3373–3385.
[13] N. Banu, D.A.S. Rees, Onset of darcy-benard convection using a thermal non-equilibrium model, Int. J. Heat Mass Transfer 45 (2002) 2221–2238.
[14] A.C. Baytas, I. Pop, Free convection in a square porous cavity using a thermal nonequilibrium model, Int. J. Therm. Sci. 41 (2002) 861–870.
[15] S.J. Kim, S.P. Jang, Effects of Darcy number, the Prandtl number, and the Reynolds number on local thermal non-equilibrium, Int. J. Heat Mass Transfer 45 (2002) 3885–3896.
[16] D.A.S. Rees, Vertical free convective boundary-layer flow in a porous medium using a thermal nonequilibrium model: elliptical effects, Z. Angew. Math. Phys. ZAMP 53 (2002) 1–12.
[17] B. Alazmi, K. Vafai, Constant wall heat flux boundary conditions in porous media under local thermal non-equilibrium conditions, Int. J. Heat Mass Transfer 45 (2002) 3071–3087.
[18] D.A. Nield, A.V. Kuznetsov, M. Xiong, Effect of local thermal non-equilibrium on thermally developing forced convection in a porous medium, Int. J. Heat Mass Transfer 45 (2002) 4949–4955.
[19] P. Vadasz, Conditions for local thermal equilibrium in porous media conduction, Transport Porous Med. 59 (2005) 341–355.
[20] J.L. Lage, The implications of the thermal equilibrium assumption for surrounding-driven steady conduction within a saturated porous medium layer, Int. J. Heat Mass Transfer 42 (1999) 477–485.
[21] M. Quintard, S. Whitaker, Local thermal equilibrium for transient heat conduction: theory and comparison with numerical experiments, Int. J. Heat Mass Transfer 38 (1995) 2779–2796.